On mixing angles and resonances in three neutrino oscillations in matter

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Abstract

We find exact analytical expressions for mixing angles in matter in the context of three generation neutrino oscillations in matter to discuss the role of resonances in this phenomenon. We show that some knowledge from conventional two neutrino MSW effect, which has been extended to approximated solutions to three neutrino oscillations, has to be abandoned in this exact approach. We observe that maximal values for the mixing angles in matter are found in nonresonant regions and stationary phases do not coincide anymore with resonances in this simple extension of the MSW effect. We present a general way to identify a resonance and discuss what we can physically expect in these regions.

I. INTRODUCTION

Resonant regions are previleged zones for neutrino conversion. Concerning solar neutrinos, the importance of a resonance can be appreciated remenbering that the standard MSW solution to the solar neutrino problem requires values for the mixing angle in vacuum θ and for the squared mass difference $\Delta = m_2^2 - m_1^2$ such that $\sin^2 2\theta < 10^{-3}$ and $\Delta \sin^2 2\theta \approx 10^{-8}$ eV^2 [1] which imply a resonance in the neutrino trajectory inside the sun when the approximately exponentially decreasing standard solar matter distribution is assumed [2]. This is the so-called nonadiabatic solution to the solar neutrino problem and the role of the resonance is evident in such situation once that it is well known that the adiabaticity parameter [3] presents its smallest values in a resonance region, which imply that neutrino transitions are less adiabatic in that region.

Resonances in two family MSW effect [3,5] are associated with maximum mixing between the two flavor eigenstates. This can be appreciated investigating the behavior of the matter mixing angle when the relevant matter density varies along the neutrino trajectory. The mixing angle in matter $\tilde{\theta}$ is introduced as the parameter that characterizes a rotation of the two-dimensional neutrino space from the basis of the current eigenstates (ν_e, ν_μ) to the basis of the physical eigenstates (ν_1, ν_2):

$$\begin{aligned} \nu_1(t) &= \nu_e(t) \cos\tilde{\theta}(t) - \nu_\mu(t) \sin\tilde{\theta}(t), \\ \nu_2(t) &= \nu_e(t) \sin\tilde{\theta}(t) + \nu_\mu(t) \cos\tilde{\theta}(t). \end{aligned} \tag{1}$$

It can be calculated [4]:

$$\sin^2 2\tilde{\theta}(t) = \frac{\sin^2 2\theta}{\left[\frac{2E\sqrt{2}G_F N_e(t)}{\Delta} - \cos 2\theta\right]^2 + \sin^2 2\theta}, \tag{2}$$

where E is the neutrino energy and $G_F N_e(t)$ is the consequence of electron neutrino coherent forward scattering from electrons in matter, the number density of which at the region reached by neutrinos at instant t is $N_e(t)$.

From Eq. (2) it is possible to see that $\tilde{\theta}$ is substantially modified by the neutrino coherent scattering from the medium. If $N_e(t) \to 0$, $\tilde{\theta} \to \theta$ and we recover vacuum expressions. When

$N_e(t)$ is extremely large, $\tilde{\theta} \to \pi/2$ and $\nu_1 \to -\nu_\mu$ while $\nu_2 \to \nu_e$. An interesting intermediate case occurs when

$$N_e(t) = \frac{1}{2\sqrt{2}G_F}\frac{\Delta}{E}\cos 2\theta \tag{3}$$

and the brackets in the denominator of Eq. (2) vanishes. In this point the mixing of flavor eigenstates is maximal, i.e., from Eq. (1) we see that the probability of finding an electron or a muon neutrino in any of the mass eigenstates is 1/2. This feature has been used to characterize a resonance: the maximum of the bell-shaped $\sin^2 2\tilde{\theta} \times N_e$ graph indicates a resonance.

The resonance condition given by Eq. (3) coincides also with the position where the difference of the two squared matter eigenvalues of the corresponding time evolution matrix in matter $\tilde{m}_2^2 - \tilde{m}_1^2$ is a minimum, suggesting that the resonance is the region where transitions between matter eigenstates are most likely to happen.

Finally, it was noticed in reference [6] that the resonance condition (3) coincides also with the condition of existence of a stationary phase [7] in the two neutrino time evolution equations. Such fact allows to investigate the evolution of this neutrino system around a resonance calculating, through the stationary phase method [7], the related Green function. Employing this method it was possible to evaluate [8] the level crossing probability, i.e., the probability of nonadiabatic transitions between matter eigenstates ν_1 and ν_2 as an alternative approach to Landau-Zener [9] or Petcov [10] methods.

In this paper we investigate how is the behavior of mixing angles in matter and how to identify a resonance in the context of a three neutrino system oscillating in matter. We assume standard electroweak interactions of neutrinos with matter as well as nonvanishing vacuum mixing angles and nondegenerated mass eigenstates (in vacuum). Therefore we are analysing the simplest extension of the conventional MSW effect [3,5] to the case where three families are present. We verify that the above mentioned three criteria usually used to define a resonance in two neutrino matter oscillations, namely, maximal mixing angles in matter, minimal eigenvalue difference and the presence of a stationary phase, do not

lead anymore to the same region in the neutrino trajectory. Note also that these same criteria have been used in approximated solutions to three neutrino oscillations in matter [11,12]. Consequently some of them have to be abandoned. We present, therefore, based on exact analytical expressions for mixing angles in matter, how we can use our previous knowledge coming from two neutrino matter oscillations to arrive to a solid condition defining resonances in three neutrino oscillations and, therefore, an accurate analytical description of the physical consequences around such regions.

II. ANALYTICAL SOLUTION

A general time evolution equation describing a three level system can be written as an equation for a three-component spinor $\Phi(t) \equiv (\Phi_1, \Phi_2, \Phi_3)$:

$$i\frac{d}{dt}\Phi(t) = h(t)\Phi(t), \tag{4}$$

where the hamiltonian $h(t)$ is a 3×3 matrix which elements are specified according to the dynamical situation from which a boundary condition $\Phi(t_o)$ is given. A general solution of Eq. (4) can be written in the the form

$$\Phi(t) = Exp\left[-i\int_{t_0}^{t} h(t')dt'\right]\Phi(t_o), \tag{5}$$

where the symbol Exp represents a sum of multiple time ordered integrals [14].

For a time-independent hamiltonian, the solution of Eq. (4) can be obtained by means of the Laplace transformation. Introducing the Laplace transformed $\Psi(p) = L\left[\Phi(t)\right]$, then

$$p\Psi(p) - \Phi(t_o) = -ih\Psi(p) \tag{6}$$

and

$$\Phi(t) = L^{-1}\left[(p\mathbf{1} + ih)^{-1}\right]\Phi(t_o). \tag{7}$$

The solution $\Phi(t)$ depends on the elements of the h matrix and on the roots λ_i $(i \equiv 1, 2, 3)$ of the characteristic polynomial of the h matrix

$$\det\left[p\mathbf{1} + ih\right] = 0. \tag{8}$$

In the particular case we are interested in, where a three neutrino system oscillates in matter, interacting with it through standard electroweak interactions, the h matrix is given by

$$h = \frac{1}{2E}\left[UM^2U^{-1} + A\right], \tag{9}$$

where M^2 is a diagonal matrix given by

$$(M^2)_{ij} = m_i^2\delta_{ij}, \tag{10}$$

m_i^2 are the three neutrino squared mass eigenvalues in vacuum,

$$U = e^{i\psi\Lambda_7}\Gamma e^{i\phi\Lambda_5}e^{i\omega\Lambda_2} \tag{11}$$

is the 3×3 mixing matrix where Λ_i are the Gell-Mann matrices, ψ, ϕ and ω are the mixing angles in vacuum and Γ is a matrix containing complex phases that we will ignore since we assume CP conservation ($\Gamma \equiv 1$).

Since we consider here only standard neutrino interactions with ordinary matter, A matrix has its first element A_{11} given by

$$A_{11} = 2\sqrt{2}G_F N_e E \tag{12}$$

and all others are zero. Note that neutral current contributions to A are proportional to the unit matrix, giving only irrelevant overall phases to the final solution of Eq. (4). G_F, E and N_e were previously introduced.

For neutrino propagating in vacuum, $A = 0$, and the solution of Eq. (4) is trivial and simply given by

$$\Phi(t) = Um^2U^{-1}\Phi(t_o), \tag{13}$$

where m^2 is a diagonal matrix with elements

$$(m^2)_{ij} = \exp\left[-i\frac{t}{2E}m_i^2\right]\delta_{ij}. \tag{14}$$

The solution of Eq. (4) in matter, with A being a time-dependent matrix, is given by Eq. (5) and it depends on the specific N_e function describing the electron density. However, when A can be considered a constant matrix, as it is supposed in the adiabatic approximation, Eq. (4) has an exact analytical solution, obtained by Laplace transformation. Furthermore, the A matrix is invariant under a $e^{i\psi\Lambda_7}$ rotation, then, introducing now

$$\Psi(t) = e^{-i\psi\Lambda_7}\Phi(t), \tag{15}$$

we observe that $\Psi(t)$ satisfies the following differential equation

$$\frac{d}{dt}\Psi(t) = -iH\Psi(t), \tag{16}$$

with boundary condition $\Psi(t_o) = e^{-i\psi\Lambda_7}\Phi(t_o)$ and

$$H = \frac{1}{2E}\left[e^{i\phi\Lambda_5}e^{i\omega\Lambda_2}M^2e^{-i\phi\Lambda_5}e^{-i\omega\Lambda_2} + A\ \right], \tag{17}$$

which can be explicitly written as

$$H = \frac{1}{4E}\begin{pmatrix} \Lambda\cos^2\phi + 2m_3^2\sin^2\phi + 2A & \Delta\sin 2\omega\cos\phi & (m_3^2 - \frac{\Lambda}{2})\sin 2\phi \\ \Delta\sin 2\omega\cos\phi & \Sigma + \Delta\cos 2\omega & -\Delta\sin 2\omega\sin\phi \\ (m_3^2 - \frac{\Lambda}{2})\sin 2\phi & -\Delta\sin 2\omega\sin\phi & \Lambda\sin^2\phi + 2m_3^2\cos^2\phi \end{pmatrix} \tag{18}$$

where $\Delta = m_2^2 - m_1^2$, $\Delta_1 = m_2^2 + m_1^2 - 2m_3^2$, $\Sigma = m_2^2 + m_1^2$ and $\Lambda = \Sigma - \Delta\cos 2\omega$.

On the Laplace space we have

$$\Psi(p) = [p\mathbf{1} + iH]^{-1}\,\Psi(t_o). \tag{19}$$

To calculate $\Psi(t)$ we have to obtain the roots of the characteristic polynomial of the matrix H, $\det[p\mathbf{1} + iH] = 0$, which are given by [15]

$$\lambda_1 = \frac{m_1^2 + m_2^2 + m_3^2 + A}{6E} - \frac{1}{E}\sqrt{\frac{-Q}{3}}\cos\frac{\alpha}{3}, \tag{20}$$

$$\lambda_2 = \frac{m_1^2 + m_2^2 + m_3^2 + A}{6E} + \frac{1}{2E}\sqrt{\frac{-Q}{3}}\cos\frac{\alpha}{3} - \frac{1}{2E}\sqrt{-Q}\sin\frac{\alpha}{3}, \tag{21}$$

$$\lambda_3 = \frac{m_1^2 + m_2^2 + m_3^2 + A}{6E} + \frac{1}{2E}\sqrt{\frac{-Q}{3}}\cos\frac{\alpha}{3} + \frac{1}{2E}\sqrt{-Q}\sin\frac{\alpha}{3}, \tag{22}$$

where

$$\alpha = \arccos\frac{-R}{2\sqrt{\frac{-Q^3}{27}}}, \tag{23}$$

$$Q = \frac{-1}{(2E)^2}\left\{\frac{\Delta^2}{4} + \frac{\Delta_1^2}{12} + \frac{A^2}{3} - \frac{A\Delta\cos^2\phi\cos 2\omega}{2} + \frac{\Delta_1 A(\cos^2\phi - 2\sin^2\phi)}{6}\right\} \tag{24}$$

and

$$\begin{aligned} R = &-\frac{1}{27(2E)^3}\left\{\frac{\Delta_1^3}{4} - 2A^3 - \frac{9\Delta^2\Delta_1}{4} + \frac{3\Delta_1^2 A(\cos^2\phi - 2\sin^2\phi)}{4}\right\} + \\ &-\frac{1}{27(2E)^3}\left\{\frac{9}{2}A^2\Delta\cos^2\phi\cos 2\omega - \frac{9}{4}A\Delta^2(\cos^2\phi - 2\sin^2\phi)\right\} + \\ &-\frac{1}{27(2E)^3}\left\{\frac{9}{2}\Delta_1 A\Delta\cos^2\phi\cos 2\omega - \frac{3}{2}A^2\Delta_1(\cos^2\phi - 2\sin^2\phi)\right\}. \end{aligned} \tag{25}$$

Note now that in vacuum we have

$$\lambda_1^v = \frac{m_3^2}{2E}, \quad \lambda_2^v = \frac{m_1^2}{2E} \quad \text{and} \quad \lambda_3^v = \frac{m_2^2}{2E}, \tag{26}$$

and the Laplace anti-transformation of Eq. (19) reproduces the corresponding solution given by Eq. (13). The roots λ_i of the characteristic polynomial are the squared mass eigenvalues in matter. Because of the arbitrariness in the choice of the order of the roots, we use the above vacuum limit to order the roots in terms of the squared mass eigenvalues in the matter. We define:

$$\lambda_1 = \frac{\tilde{m}_3^2}{2E}, \quad \lambda_2 = \frac{\tilde{m}_1^2}{2E} \quad \text{and} \quad \lambda_3 = \frac{\tilde{m}_2^2}{2E}. \tag{27}$$

Finally, we can write the solution of Eq. (4) in terms of a T transition matrix such that

$$\Phi(t) = e^{i\psi\Lambda_7} T e^{-i\psi\Lambda_7}\Phi(t_0), \tag{28}$$

where the elements of the T matrix, given in terms of the λ_i roots and of the elements of the H matrix, can be written as:

i) diagonal elements:

$$T_{ii} = \sum_{m=1}^{3} C_m \left[(\lambda_m - H_{jj})(\lambda_m - H_{kk}) - H_{jk}^2\right] e^{-i\lambda_m t}, \tag{29}$$

ii) non diagonal elements ($T_{ij} = T_{ji}$):

$$T_{ij} = \sum_{m=1}^{3} C_m \left[H_{ij}(\lambda_m - H_{kk}) - H_{ik}H_{jk}\right] e^{-i\lambda_m t}, \tag{30}$$

where

$$C_m = \left[(\lambda_m - \lambda_\ell)(\lambda_m - \lambda_n)\right]^{-1} \tag{31}$$

with $m \neq \ell \neq n$ and $n, \ell, n \equiv (1, 2, 3)$.

Note also that all well known results for a two neutrino system oscillating in matter can be straightforwardly obtained from the solution given by Eq. (28).

III. MIXING ANGLES IN MATTER

It is well known that the knowledge of the mixing angles in matter is important to study resonant transitions between flavor neutrino states [4]. In order to explicitly write an exact expression for these angles, we define $\tilde{\psi}, \tilde{\phi}$ and $\tilde{\omega}$ as the mixing angles in the matter. We can write therefore the final solution of Eq. (4) in terms of mixing angles in matter in analogy with what we did in the vacuum case, Eq.(13), using now the final solution given by Eq. (28). This solution can be written in the following way

$$\Phi(t) = U(\tilde{\psi}, \tilde{\phi}, \tilde{\omega}) M_m^2 U^{-1}(\tilde{\psi}, \tilde{\phi}, \tilde{\omega}) \Phi(t_0) \tag{32}$$

where

$$(M_m^2)_{ij} = \exp[-it\lambda_i]\delta_{ij} \tag{33}$$

and

$$U(\tilde{\psi}, \tilde{\phi}, \tilde{\omega}) = e^{i\tilde{\psi}\Lambda_7} e^{i\tilde{\phi}\Lambda_5} e^{i\tilde{\omega}\Lambda_2}. \tag{34}$$

In order to get the matter mixing angles we simply compare Eq. (32) with Eq. (28), and after some algebra, we obtain

$$\sin^2\tilde{\phi} = \frac{\lambda_1^2 - (H_{22}+H_{33})\lambda_1 + H_{22}H_{33} - H_{23}^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \tag{35}$$

$$\tan^2\tilde{\omega} = \frac{[\lambda_3^2 - (H_{22}+H_{33})\lambda_3 + H_{22}H_{33} - H_{23}^2](\lambda_1-\lambda_2)}{[\lambda_2 2 - (H_{22}+H_{33})\lambda_2 + H_{22}H_{33} - H_{23}^2](\lambda_3-\lambda_1)} \tag{36}$$

and

$$\tan\tilde{\psi} = \frac{(H_{12}\lambda_1 + H_{13}H_{23} - H_{12}H_{33})\cos\psi + (H_{13}\lambda_1 + H_{12}H_{23} - H_{22}H_{13})\sin\psi}{(H_{13}\lambda_1 + H_{12}H_{23} - H_{22}H_{13})\cos\psi - (H_{12}\lambda_1 + H_{13}H_{23} - H_{12}H_{33})\sin\psi}. \tag{37}$$

IV. ANALYSIS OF THE RESULTS

In Fig. 1 it is presented a comparison of the behavior of the quadratic matter eigenvalues $\tilde{m}_i^2$ and the relevant matter mixing angles $\tilde{\omega}$ and $\tilde{\phi}$ as a function of the parameter A for specially chosen values of vacuum parameters (see the corresponding caption for details. A is given in units of m_1^2). There are two resonances clearly indicated by the minimum difference between the shown quadratic masses. We observe that $\tilde{\omega}$ presents a maximum value in the lower resonance (where $\tilde{m}_2^2 - \tilde{m}_1^2$ is minimum) while $\tilde{\phi}$ shows a maximum in the region of the higher resonance ($\tilde{m}_3^2 - \tilde{m}_2^2$ is a minimum). Interesting enough, differently from what is expected in the two flavor neutrino oscillations in matter, the conventional MSW effect, and also from what was found in previous approximated analyses of the three neutrino oscillations [11,12], a second peak for the mixing angle $\tilde{\omega}$ is found after the higher resonance [13]. In Fig. 2 we show the same graphs presented in Fig. 1 to evidenced this unexpected behavior of the mixing angle $\tilde{\omega}$ for larger values of A. It is clear from this figure that the criterion of defining a resonance by means of localizing the maximal mixing angle in matter, which can be safely used in two neutrino conventional MSW effect, leads to some ambiguity in the context of three neutrino oscillations and therefore has to be abandoned.

Instead, we can improve this criterion analysing the content of Figs. 3 and 4. Note that the admixture of flavor eigenstates in each of the matter eigenstates can be obtained through

$\tilde{\nu}_i = \sum_\alpha U_{i\alpha}\nu_\alpha$, where $i = 1, 2, 3$, $\alpha = e, \mu, \tau$ and $U_{i\alpha}$ is given by Eq. (11). Let us write now, as an example, the linear combination of flavor eigenstates in the first matter eigenstates:

$$\tilde{\nu}_1 = \cos\tilde{\phi}\cos\tilde{\omega}\nu_e + \cos\tilde{\phi}\sin\tilde{\omega}\nu_\mu + \sin\tilde{\phi}\nu_\tau. \tag{38}$$

In Figs. 3 and 4 we show therefore the coefficients of this admixture (values of the vacuum parameters are shown in the corresponding captions). From Fig. 3 we observe that in the lower resonance the mixing of electronic and muonic flavor eigenstates is maximal (when $\cos\tilde{\phi}\cos\tilde{\omega} = \cos\tilde{\phi}\sin\tilde{\omega}$), while, from Fig. 4, we see that the higher resonance coincides with the maximum admixture of ν_μ and ν_τ, when $\cos\tilde{\phi}\sin\tilde{\omega} = \sin\tilde{\phi}$.

Therefore, although we detected maxima of the mixing angles in matter in regions far from resonances, it is still possible to identify a resonance region searching for maximal mixing between flavor eigenstates. Note also that such maximum are not anymore related with values of $\sqrt{2}/2$ for flavor coefficients $|U_{i\alpha}|$ in the way it happened in the conventional MSW phenomenon. This is because there could be nonnegligible contributions from the flavor eigenstate that does not participate in the resonant process. From the unitarity of the mixing matrix, we know that $\tilde{U}_{ie}|^2 + |\tilde{U}_{i\mu}|^2 + |\tilde{U}_{i\tau}|^2 = 1$, for $i = 1, 2, 3$. Therefore, in the case where $i = 1$ and $|\tilde{U}_{1\tau}|$ is not vanishing, the maximal mixing is such that $|\tilde{U}_{1e}| = |\tilde{U}_{1\mu}| < \sqrt{2}/2$. A similar situation occurs for the higher resonance where we obtain $|\tilde{U}_{1\mu}| = |\tilde{U}_{1\tau}| < \sqrt{2}/2$. We can say that in three neutrino oscillation phenomenon the mixing between flavor eigenstate around a resonant region is as maximal as possible, although not in the same way as in two neutrino oscillations, where maximum mixing implies that each one of the neutrino flavor eigenstate participating in the resonant process contributes with 50% to the matter eigenstates.

A final issue to be discussed is the criterion of identifying a resonance looking for a stationary phase in the neutrino evolution equations (4), in the same way it was proposed in reference [6] in the context of two neutrino MSW effect. A stationary phase is given by the smallest difference of any two diagonal elements of the relevant evolution matrix when one of these elements is time dependent. As an example, we quote solar neutrinos where

the matter density considerably varies along the neutrino trajectory from the center of the sun, where neutrinos are created, to the solar surface. Although in two neutrino oscillations this criterion can be safely used, it does not work anymore in the presently analysed three neutrino MSW effect. Stationary phases do not coincide with the minimum squared mass differences or maximum flavor admixture.

Note however that it is still possible to use the stationary phase method to calculate level crossing probabilities in the three neutrino oscillations. Making convenient SU(3) transformations on the evolution matrix (9) it is possible to conciliate resonances and stationary phases. This is because resonances are invariant under similarity transformations, while stationary phases do not. Therefore the matrix

$$H^1 = e^{-i\phi\Lambda_5} H e^{i\phi\Lambda_5} \tag{39}$$

presents a stationary phase for the minimum of $H^1_{11} - H^1_{22}$ coinciding with the minimum of the squared mass difference $\tilde{m}_2^2 - \tilde{m}_1^2$, and it can be used to calculate the level crossing probability [8] around the lower resonance. To obtain the correct stationary phase to analyse the higher resonance, we rotate the evolution matrix given in Eq. (4) in the following way:

$$H = e^{-i\psi\Lambda_7} h e^{i\psi\Lambda_7} \tag{40}$$

Now the minimum of the difference $H_{33} - H_{11}$ indicates a stationary phase which now coincides with the required resonance.

V. CONCLUSIONS

Resonances represent a crucial region in the time evolution of neutrinos oscillating in matter. They are closely related with the nonadiabatic character of the oscillation. We investigated a general criterion to define a resonant region when three neutrino are present in the oscillation phenomenon. We observed that two of the three commonly employed criteria to identify a resonance in two neutrino oscillations are not valid anymore in its

simplest extension to three neutrino MSW effect. For instance, mixing angles can present maximal values far from resonant regions and therefore this criterion to define a resonance has to be abandoned. Furthermore, stationary phases do not necessarily coincides with resonant regions. The safest way to identify such resonance regions is to investigate the behavior of the squared matter eigenvalue differences, looking for their minimum values.

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CAPTIONS

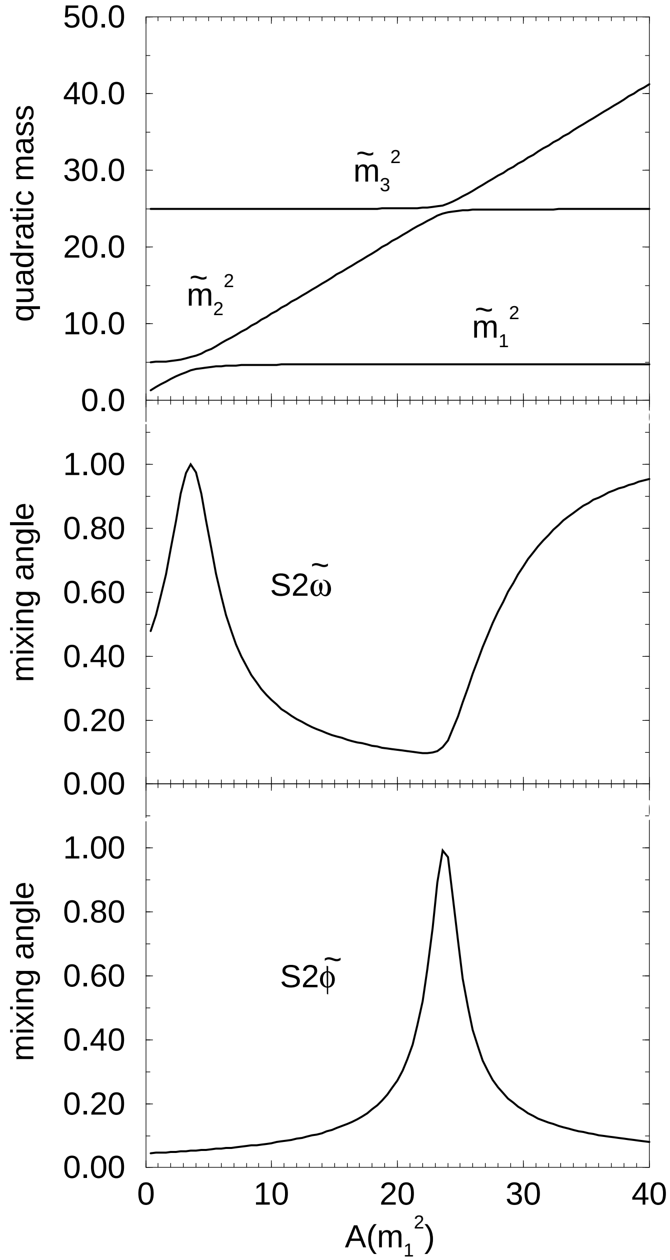
Figure 1: Squared matter eingenvalues $\tilde{m}_i^2$ and the relevant matter mixing angles $\tilde{\omega}$ and $\tilde{\phi}$ as a function of the parameter A are presented. The values of the parameters ϕ, ω, and m_i^2 in vacuum: $m_3^2 = 5m_2^2 = 25m_1^2$; $\sin^2\phi = 5\times10^{-4}$; $\sin^2\omega\cos^2\phi = 5\times10^{-2}$ were chosen in order to well demonstrate the behavior of these parameters as a function of A. The eingenvalues $\tilde{m}_i$ and the quantity A are given in units of m_1^2.

Figure 2: Mixing angles $\tilde{\omega}$ and $\tilde{\phi}$ as a function of the A for larger values of A. The parameters ϕ, ω, and m_i^2 are the same as that of the Figure 1.

Figure 3: The squared mass difference $\tilde{m}_2^2 - \tilde{m}_1^2$, and the quantities $\cos\tilde{\phi}\sin\tilde{\omega}$ and $\cos\tilde{\phi}\cos\tilde{\omega}$ quantities are presented as a function of the energy of neutrinos. The values of the parameters ϕ, ω and m_i^2 in vacuum are: $m_3^2 = 1.445\times10^{-4}eV^2$; $m_2^2 = 10^{-8}\mathrm{eV}^2$; $m_1^2 = 0$; $\sin^2\phi = 5\times10^{-4}$; $\sin^2\omega = 0.050025$. The squared masses are given in units of m_2^2.

Figure 4: The squared mass difference $\tilde{m}_3^2 - \tilde{m}_2^2$, the effective $\cos\tilde{\phi}\sin\tilde{\omega}$ and $\sin\tilde{\phi}$ quantities are presented as a function of the energy of neutrinos. The values of the parameters ϕ , ω and m_i^2 in vacuum are the same ones used to draw Fig. 3.

Figure 1



50.0
40.0
30.0
20.0
10.0
0.0
quadratic mass
$\tilde{m}_3^2$
$\tilde{m}_2^2$
$\tilde{m}_1^2$
1.00
0.80
0.60
0.40
0.20
0.00
mixing angle
$S2\tilde{\omega}$
1.00
0.80
0.60
0.40
0.20
0.00
mixing angle
$S2\tilde{\phi}$
0
10
20
30
40
$A(m_1^2)$

Figure 2

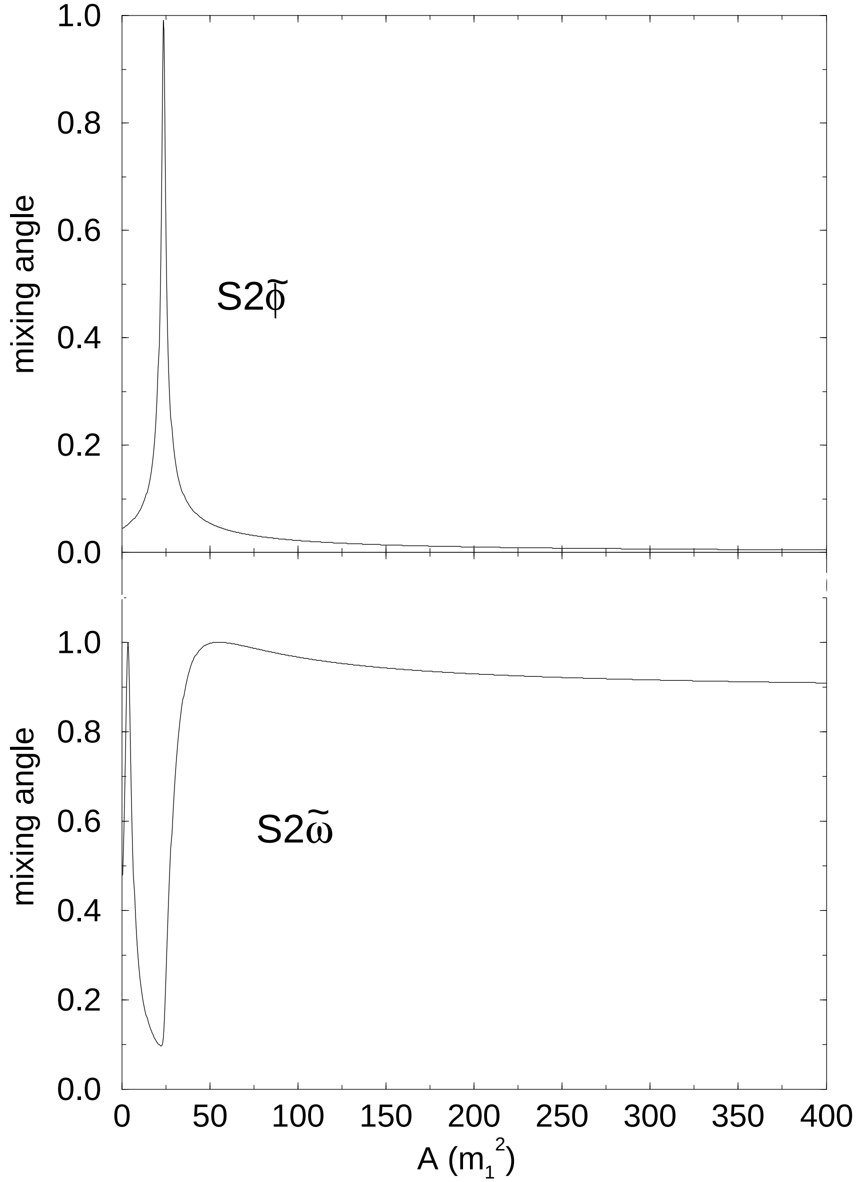
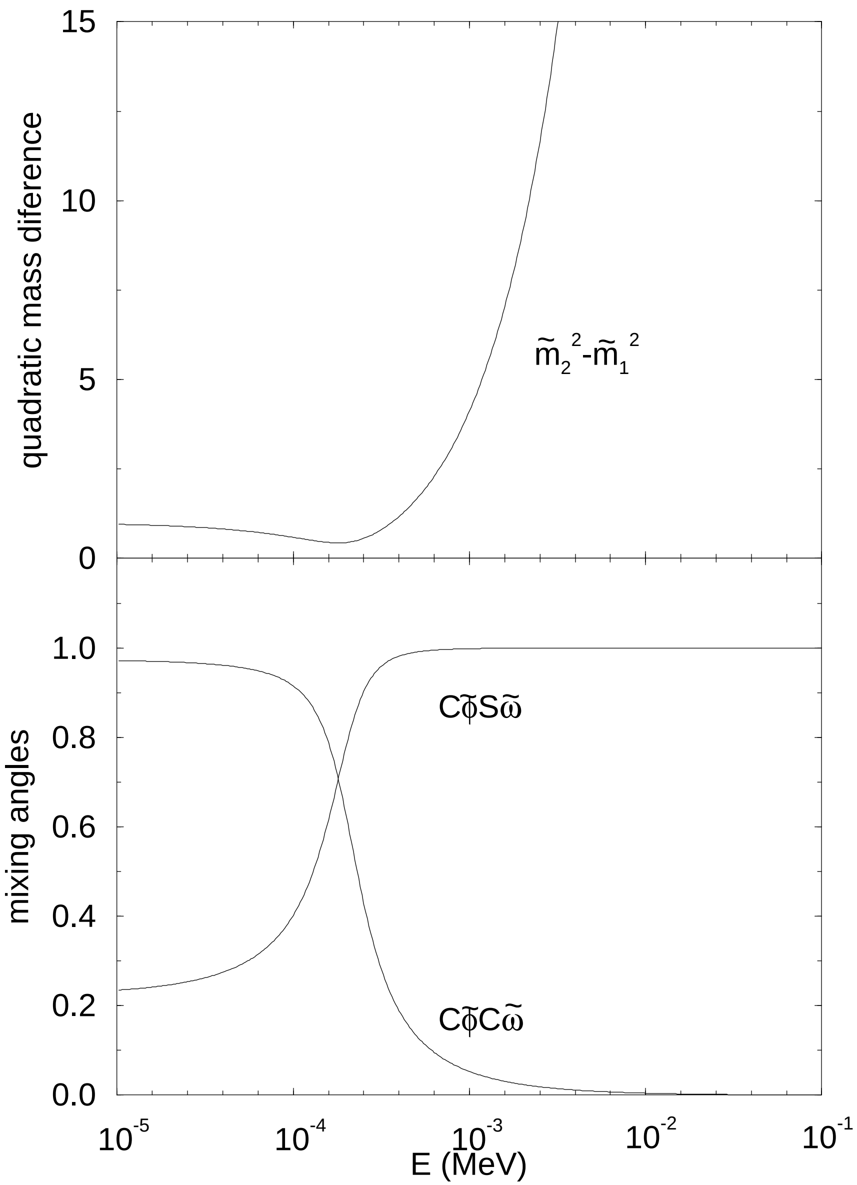
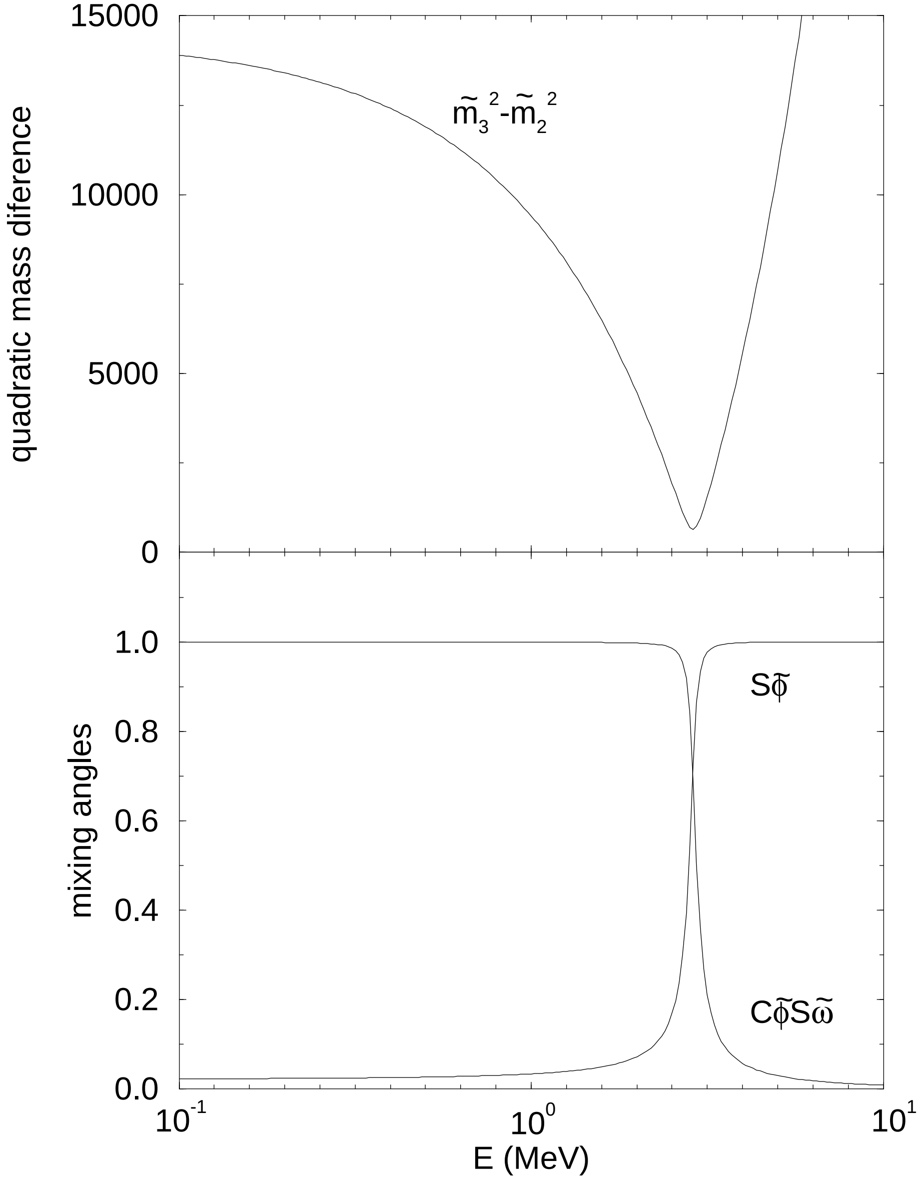

mixing angle
S2$\tilde{\phi}$
mixing angle
S2$\tilde{\omega}$
A (m_1^2)
1.0
0.8
0.6
0.4
0.2
0.0
0
50
100
150
200
250
300
350
400

Figure 3



15
10
5
0
quadratic mass diference
$\tilde{m}_2^2-\tilde{m}_1^2$
1.0
0.8
0.6
0.4
0.2
0.0
mixing angles
$C\tilde{\phi}S\tilde{\omega}$
$C\tilde{\phi}C\tilde{\omega}$
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
E (MeV)

Figure 4



15000
10000
5000
0
quadratic mass diference
$\tilde{m}_3^2-\tilde{m}_2^2$
1.0
0.8
0.6
0.4
0.2
0.0
mixing angles
$S\tilde{\phi}$
$C\tilde{\phi}S\tilde{\omega}$
10^{-1}
10^{0}
10^{1}
E (MeV)